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Generalized point operators for pythagorean fuzzy information and their application in multi-criteria decision making

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Abstract. Pythagorean fuzzy sets, which is based on intuitionistic fuzzy sets (IFSs), is an important tool to solve problems and has attached a large number of researchers in different fields. As we know, studies have focused on interval-valued Pythagorean fuzzy set and aggregated operators. However, few studies focus on point operators. This paper introduces and discusses what is the pythagorean fuzzy point operators, study their properties and relationships, which is seen as the extensions of intuitionistic fuzzy sets. The uncertainty regarding to Pythagorean fuzzy set could be decreased if we use the pythagorean fuzzy point operators. Also, an approach has been introduced to address multi-attributes decision making issues based on pythagorean fuzzy sets.

Key words. Point operators, multi-criteria decision making, pythagorean fuzzy set.

1. Introduction

Many new means and theories have been brought forward since Zadeh[1] developed the fuzzy set[2-6]. Based on this, intuitionistic fuzzy set was introduced by Atanassov[7] which led researchers investigating more meaningful conclusions and applied it to resolve issues especially in multicriteria decision making. In recent years, the definition of Pythagorean fuzzy set (PFS) has been discussed by Yager[8] and seen as an important expansion of the intuitionistic fuzzy sets.

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Since it appeared, many scholars have studied it and got a lot of achievements about PFS[9-11]. For example, Yager[8] introduced some new fuzzy weighted average and geometric aggregated operators to treat Pythagorean fuzzy MADM issues. Zhang and Xu proposed a method to discover the best alternative by using the ideal plan under the Pythagorean fuzzy environment. VahidMohagheghi offers the newest procedure of a novel polymerization group decision-making, and it can be used to weigh and evaluate data. This method is very flexible and accurate when there is a big difference between judgement of makers.

Many studies focus on the interval-valued intuitionistic and pythagorean fuzzy set and aggregated operators to solve multicriteria decision making. However, few studies focus on point operators under Pythagorean fuzzy environment. So this paper proposes this theory for the purpose to decrease uncertain information and improve the accuracy of information.

In this paper, the definition of intuitionistic fuzzy sets and pythagorean fuzzy sets are first reviewed and some defined operations for PFS are introduced. Then some related concepts based on intuitionistic fuzzy point operators are given. Further, an in-depth study on the point operators is given, discover some meaningful results and put forward some new ideas.

2. Perlimininaries

Definition 1 An intuitionistic fuzzy set (IFS) Lon a fixed set X is delimited as

$$L = \{\langle x, \mu_L(x), \nu_L(x)\rangle \,|x \in X\} \tag{1}$$

with the condition that $0 \leq \mu_L(x) + v_L(x) \leq 1$, $\mu_L(x) \geq 0$ and$v_L(x) \geq 0$, the degree of hesitation is denoted by $\pi_L(x) = 1 - \mu_L(x) - v_L(x)$.

Pythagorean fuzzy set (PFS), which was discussed by Yager [8], can be defined as:

Definition 2 Assume X is a fixed set. We call the triad L as pythagorean fuzzy set (PFS):

$$L=\{\langle x, \mu_L(x), \nu_L(x)\rangle \,|x \in X\} \tag{2}$$

In which the function $\mu_L : X \to [0,1]$ delimits the membership and $\nu_L : X \to [0,1]$ delimits the non-membership which based on the element $x \in X$ of L. For every $x \in X$,it requires the condition that $(\mu_L(x))^2 + (\nu_L(x))^2 \leq 1$. The degree of hesitant-ion is given by $\pi_L(x) = \sqrt{1 - (\mu_L(x))^2 - (\nu_L(x))^2}$.

Definition 3 For any two PFNs L_1, L_2, let $S(L_i)$ be the score value of L_i, then

If $S(L_1) \prec S(L_2)$, then $L_1 \prec L_2$

If $S(L_1) \succ S(L_2)$, then $L_1 \succ L_2$

If $S(L_1) = S(L_2)$, then $L_1 \sim L_2$

Where $S(L) \in [-11]$.

The accuracy function of L can be delimited as$H(L) = (\mu_L)^2 + (\nu_L)^2$in which $H(L) \in [01]$.

Definition 4 Assume that m dimension of the GOWA operator can be cast light upon GOWA:$H^m \to H$ and owns the shape below:

$$GOWA\left[\gamma_1,\gamma_2...\gamma_m\right]=\left(\sum_{j=1}^{m}\omega_j b_j^{\eta}\right)^{\frac{1}{\eta}} \quad (3)$$

Where$\eta \in [-\infty+\infty]$,ω= $(\omega_1\omega_2...,\omega_m)^T$ is the relational weighting vector with$\omega_j \geq 0$, j=1,2,. . . ,m, $\sum_{j=1}^{m}\omega_j = 1$, b_j is the jth biggest of γ_i(i=1,2,. . . ,m), H=[0,1].

Definition 5 Assume that there is an IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, let $\kappa\lambda \in [0, 1]$, Atanassov presented the following operators:

$$D_\kappa(A) = \{x, \langle\mu_A(x) + \kappa\pi_A(x), \nu_A(x) + (1-\kappa)\pi_A(x)\rangle | x \in X\} \quad (4)$$

$$F_{\kappa\lambda}(A) = \{x, \langle\mu_A(x) + \kappa\pi_A(x), \nu_A(x) + \lambda\pi_A(x)\rangle | x \in X\} \quad (5)$$

where $\kappa+\lambda \leq 1$.

Note IFS(X) as the set of all IFSs on X. For $A \in IFS(x)$??an operator $D_{\kappa_x}(A)$ is defined for each $x \in X$:

$$D_{\kappa_x}(A) = \{x, \langle\mu_A(x) + \kappa_x\pi_A(x), \nu_A(x) + (1-\kappa_x)\pi_A(x)\rangle | x \in X\} \quad (6)$$

Where $\kappa_x \in [0, 1]$.

Definition 6 For an IFV α= $(\mu_\alpha\nu_\alpha)$, let $\kappa_\alpha, \lambda_\alpha \in [0, 1]$, some point operators: IFV IFV are delimited as:

$$D_{\kappa_\alpha}(\alpha) = (\mu_\alpha + \kappa_\alpha\pi_\alpha\nu_\alpha + (1-\kappa_\alpha)\pi_\alpha) \quad (7)$$

$$F_{\kappa_\alpha,\lambda_\alpha}(\alpha) = (\mu_\alpha + \kappa_\alpha\pi_\alpha\nu_\alpha + \lambda_\alpha\pi_\alpha), \quad (8)$$

where $\kappa_\alpha+\lambda_\alpha \leq 1$.

Point operators for aggregating PFS

A new concept of point operators for PFN is developed in this section and some examples are given.

Definition 7 For a pythagorean fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, let $\kappa\lambda \in [0, 1]$, then

$$D_\kappa(A) = \left\{x, \left\langle\mu_A^2(x) + \kappa\pi_A^2(x), \nu_A^2(x) + (1-\kappa)\pi_A^2(x)\right\rangle | x \in X\right\} \quad (9)$$

$$F_{\kappa\lambda}(A) = \left\{x, \left\langle\mu_A^2(x) + \kappa\pi_A^2(x), \nu_A^2(x) + \lambda\pi_A^2(x)\right\rangle | x \in X\right\} \quad (10)$$

where $\kappa+\lambda \leq 1$.

Proof.

Let

$$x = \mu_A^2(x) + \kappa\pi_A^2(x),$$

$$y = \nu_A^2(x) + (1-\kappa)\pi_A^2(x),$$
$$x+y = \mu_A^2(x) + \nu_A^2(x) + \pi_A^2(x) = 1$$

Then

$0 \le x \le 1, 0 \le y \le 1$, so we have $x^2 + y^2 \le 1$ satisfy the conditions.

Definition 8 For a pythagorean fuzzy number $\alpha{=}(\mu_\alpha \nu_\alpha)$, let $\kappa_\alpha, \lambda_\alpha \in [0,1]$, some new point operators: $PFV \to PFV$ are defined as:

$$D_{\kappa_\alpha}(\alpha) = \left(\mu_\alpha^2 + \kappa_\alpha \pi_\alpha^2, \nu_\alpha^2 + (1-\kappa_\alpha)\pi_\alpha^2\right)$$
$$F_{\kappa_\alpha \lambda_\alpha}(\alpha) = \left(\mu_\alpha^2 + \kappa_\alpha \pi_\alpha^2, \nu_\alpha^2 + \lambda_\alpha \pi_\alpha^2\right)$$

where $\kappa_\alpha + \lambda_\alpha \le 1$.

Theorem 1. Let $A \in PFS(\partial)$, $\alpha \in \partial$, $\kappa_\alpha, \lambda_\alpha \in [0,1]$ and $\kappa_\alpha + \lambda_\alpha \le 1$.

If $\mu_A^2(\alpha) + \nu_A^2(\alpha) \neq 0$ for all $\alpha \in \partial$ and $\kappa_\alpha = \frac{\mu_A^2(\alpha)}{\mu_A^2(\alpha)+\nu_A^2(\alpha)}$ and $\lambda_\alpha = \frac{\nu_A^2(\alpha)}{\mu_A^2(\alpha)+\nu_A^2(\alpha)}$,

Then $F_{\kappa_\alpha \lambda_\alpha}(A) = \{(\alpha, \kappa_\alpha \lambda_\alpha) \,|\, \alpha \in \partial\}$.

Definition 9 Assume that $\alpha_j{=}\left(\mu_{\alpha_j} \nu_{\alpha_j}\right)$(j=1,2,. . . ,n) is a set of PFVs. nis a plus integer, $\kappa_{\alpha_j}, \lambda_{\alpha_j} \in [0,1]$, $\omega{=}(\omega_1 \omega_2 ..., \omega_n)^T$ is a w-eight vector corresponding to PFWAD(F) operators, $\omega_j \ge 0$ and $\sum_{j=1}^n \omega_j = 1$, Then the PFWAD (F) aggre-gation value is also PFV, and

$$PFWAD(\alpha_1 \alpha_2, ..., \alpha_n) = \omega_1 \left(D_{\kappa_{\alpha_1}}(\alpha_1)\right) \oplus \omega_2 \left(D_{\kappa_{\alpha_2}}(\alpha_2)\right) \oplus ... \oplus \omega_n \left(D_{\kappa_{\alpha_n}}(\alpha_n)\right)$$

$$= \left\langle \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\mu_{\alpha_j}^2 + \kappa_{\alpha_j} \pi_{\alpha_j}^2\right)^2\right)^{\omega_j}}, \prod_{j=1}^{n} \left(\nu_{\alpha_j}^2 + \left(1 - \kappa_{\alpha_j}\right) \pi_{\alpha_j}^2\right)^{\omega_j} \right\rangle \quad (11)$$

$$PFWAF(\alpha_1 \alpha_2, ..., \alpha_n) = \omega_1 \left(F_{\kappa_{\alpha_1}}(\alpha_1)\right) \oplus \omega_2 \left(F_{\kappa_{\alpha_2}}(\alpha_2)\right) \oplus ... \oplus \omega_n \left(F_{\kappa_{\alpha_n}}(\alpha_n)\right)$$

$$= \left\langle \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\mu_{\alpha_j}^2 + \kappa_{\alpha_j} \pi_{\alpha_j}^2\right)^2\right)^{\omega_j}}, \prod_{j=1}^{n} \left(\nu_{\alpha_j}^2 + \lambda_{\alpha_j} \pi_{\alpha_j}^2\right)^{\omega_j} \right\rangle \quad (12)$$

3. An approach to pythagorean fuzzy multi-attributes decision making

The following part is mainly around the Pythagorean fuzzy sets of multi-attributes decision making. Let $y = \{y_1, y_2, \cdots, y_m\}$ be some alternatives to be selected, and

$c = \{c_1, c_2, \cdots, c_n\}$ be some criteria to be evaluated. The performance of y_i under the criteria c_j can be noted as an PFN $\alpha_{ij} = (\mu_{ij}, v_{ij})$ with the condition $0 \leq \mu_{ij}, v_{ij} \leq 1$and $\mu_{ij}^2 + v_{ij}^2 \leq 1$. Give the following steps to get the best proposal:

Step 1. Aggregate the pythagorean fuzzy values β_i of the alternative $y_i(i = 1, 2, ..., m)$, by the PFWAD operator:

$$\beta_i = PFWAD\left(\alpha_1\alpha_2, ..., \alpha_n\right) = \omega_1\left(D_{\kappa_{\alpha_1}}\left(\alpha_1\right)\right) \oplus \omega_2\left(D_{\kappa_{\alpha_2}}\left(\alpha_2\right)\right) \oplus ... \oplus \omega_n\left(D_{\kappa_{\alpha_n}}\left(\alpha_n\right)\right)$$

$$= \left\langle \sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\mu_{\alpha_j}^2 + \kappa_{\alpha_j}\pi_{\alpha_j}^2\right)^2\right)^{\omega_j}},\ \prod_{j=1}^{n}\left(\nu_{\alpha_j}^2 + \left(1 - \kappa_{\alpha_j}\right)\pi_{\alpha_j}^2\right)^{\omega_j} \right\rangle \quad (13)$$

Step 2. Compute the scoring values $s(\alpha_i)$ of α_i by Definition 3, and the best proposal can be obtained according to the ranking of α_i $(i = 1, 2, \cdots, m)$, the bigger the α_i represents the better the alternative y_i.

Step 3. Conclude the bigger one which is the best choice.

So as to demonstrate the method put forward above, an example is given as follows:

Example. A unit in the cadre selection made four evaluation indexes: G_1: morality, G_2: style of work, G_3: educational level and G_4: leadership ability. The weight vector of the indexes is ω= $(0.15, 0.25, 0.35, 0.25)$, k=0.3. After recommendation, three candidates T_i $(i = 1, 2, 3)$ are determined. Assumed that the assessment information of each candidate is shown as table 1, we need to choose the best candidate.

Table 1. Pythagorean fuzzy decision matrix

	G_1	G_2	G_3	G_4
T_1	(0.4,0.7)	(0.9,0.2)	(0.8,0.1)	(0.5,0.3)
T_2	(0.8,0.4)	(0.7,0.5)	(0.6,0.2)	(0.7,0.4)
T_3	(0.7,0.2)	(0.8,0.2)	(0.8,0.4)	(0.6,0.6)

Step 1. Aggregate the pythagorean fuzzy values α_i of the alternatives G_i by the PFWAD operator of each candidate.

Step 2. Calculate the scores $s(\alpha_i)$ of α_i by Definition 3:

$$s(\alpha_1) = 0.3351, s(\alpha_2) = 0.1125, s(\alpha_1) = 0.2572$$

Step 3. Since $s(\alpha_1)>s(\alpha_3)>s(\alpha_2)$, we can conclude that: $X_1>X_3>X_2$, so candidateX_1 is the best choice.

In fact, whatever k takes, we can always draw the same conclusion.

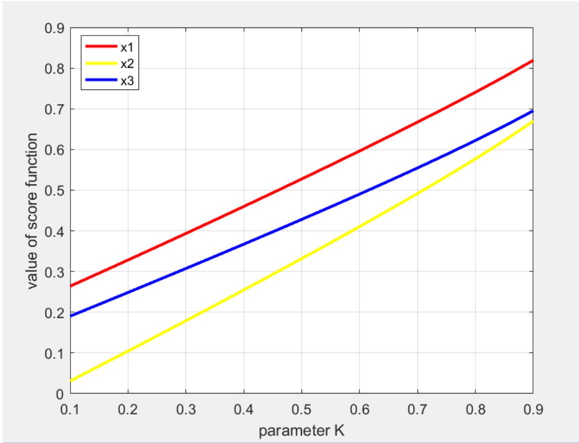

Fig. 1. Value of score function obtained by the PFWAD operator

We use PFWA operator as contrast:

$$PFWA=\left(\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{2}\right)^{\omega_j}},\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j}\right) \tag{14}$$

By using Eq.(23), the pythagorean fuzzy values α_i of the alternatives are α_1=(0.762,0.2096), α_2=(0.6899,0.3318), α_3=(0.7497,0.3355). Then we can compute the score values similarly: $s(\alpha_1)$=0.5524, $s(\alpha_2)$=0.3581, $s(\alpha_3)$=0.4142.

The result is $X_1>X_3>X_2$. This indicates that the consequence is consistent and the method we proposed is feasible.

4. Conclusion

This paper gives a further study about the pythagorean fuzzy set, and develops a series of new point operators for PFNs, study the attributes and relevance of them. PFS have been further discussed, and some important conclusions have been obtained. Moreover, a solution is proposed to deal with multi-attribute decision making problems using point operators. A case is given to illustrate that this method is more effective than PFWA aggregation.

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